Effects of Force and Energy in an Evolving Universe with a Gravitational Source

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Abstract

Two models are given by crossing the Friedmann metrics with Schwarzschild and Kerr metrics. In these evolving universes with a gravitational source, the force four-vector and the corresponding potentials are evaluated.

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1 Introduction

The Friedmann models are without gravitational source due to their homogeneity. There are models which combine the Friedmann universe with a Schwarzschild metric [1], but only one metric acts at any point in the spacetime. There being no model which could show both the expansion of the universe and a gravitational attraction together. Bokhari and Qadir [2] presented an alternative way of constructing a toy model which gives an effect of a gravitational source in an evolving universe. The spatial part of this metric is the same as the Schwarzschild metric, but multiplied by a time dependent scale factor. The time component is the usual Schwarzschild time component

$$ds^2 = e^{\nu(r)}dt^2 - a^2(t)[e^{-\nu(r)}dr^2 + r^2 d\Omega^2], \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi$ is the solid angle. This metric admits of a conformal time-like Killing vector. It provides a gravitational source for the flat Friedmann model and cannot be extended to the open and closed Friedmann universes. The force was evaluated by using the conformal pseudo-Newtonian ($c\psi N$)-formalism [2] which deals with conformally static spacetimes. The time component of the force four-vector does not appear there. Thus the metric and the $c\psi N$-formalism has weakness of its own. In this paper we construct a new metric which is applicable to all the three (flat, open, closed) Friedmann models. Further we take another metric which is a cross between the Kerr and the Friedmann metrics only for flat case. We evaluate the force four-vector and energy imparted to a test particle for these metrics so as to be able to analyse them in terms of forces and energy. To this end we use the extended ($e\psi N$)-formalism which deals with non-static spacetimes explicitly. We will not discuss the $e\psi N$-formalism in any detail as that is discussed separately [3,4].

The plan of the paper is as follows. In the next section we shall briefly review the essential points of the $e\psi N$-formalism for the purposes of application. In section three we determine the forces and energy for the Friedmann-Schwarzschild crossed metric and analyse them. In the next section we evaluate the $e\psi N$ force and potential for the Friedmann-Kerr crossed metric only for the flat case. Finally in the fifth section the results are summarised and discussed.

2 The $e\psi N$-Formalism

The basis of the formailsm is the observation that the tidal force, which is operationally determinable, can be related to the curvature tensor by

$$F_T^\mu = mR^\mu_{\nu\rho\pi}t^\nu l^\rho t^\pi, \quad (\mu,\nu,\rho,\pi = 0,1,2,3), \tag{2}$$

where m is the mass of a test particle, $t^\mu = f(x)\delta^\mu_0, \quad f(x) = (g_{00})^{-1/2}$ and l^μ is the separation vector. l^μ can be determined by the requirement that the tidal force have maximum magnitude in the direction of the separation vector. Choosing a gauge in which $g_{0i} = 0$ (similar to the synchronous coordinate system [5,6]) in a coordinate basis. We further use Riemann normal coordinates (RNCs) for the spatial direction, but not for the temporal direction. The reason for this difference is that both ends of the accelerometer are spatially free, i.e. both move and do not stay attached to any spatial point. However, there is a "memory" of the initial time built into the accelerometer in that the zero position is fixed then. Any change is registered that way. Thus "time" behaves very differently from "space".

The $e\psi N$ force, F_μ, satisfies the equation

$$F_T^{*\mu} = l^\nu F^\mu_{;\nu}, \tag{3}$$

where $F_T^{*\mu}$ is the extremal tidal force corresponding to the maximum magnitude reading on the dial. Notice that $F_T^{*0} = 0$ does not imply that $F^0 = 0$. The requirement that Eq.(3) be satisfied can be written as

$$l^i(F^0_{,i} + \Gamma^0_{ij}F^j) = 0, \tag{4}$$

$$l^j(F^i_{,j} + \Gamma^i_{0j}F^0) = F_T^{*i}. \tag{5}$$

A simultaneous solution of the above equations can be found by taking the ansatz

$$F^0 = m\left[(\ln A)_{,0} - \Gamma^0_{00} + \Gamma^i_{0j}\Gamma^i_{0i}/A\right]f^2, \tag{6}$$

$$F^i = m\Gamma^i_{00}f^2, \tag{7}$$

where $A = (\ln\sqrt{-g})_{,0}, \quad g = det(g_{ij})$. These equations can be written in terms of two quantities U and V given by

$$U = m\left[\ln(Af/B) - \int(g^{ij}_{,0}g_{ij,0}/4A)dt\right], \tag{8}$$

$$V = -m \ln f, \tag{9}$$

as

$$F_0 = -U_{,0}, \quad F_i = -V_{,i}. \tag{10}$$

It is to be noted that the momentum four-vector p_μ can be written in terms of the integral of the force four-vector F_μ. Thus

$$p_\mu = \int F_\mu dt. \tag{11}$$

Notice that the zero component of the momentum four-vector corresponds to the energy imparted to a test particle of mass m while the spatial components give the momentum imparted to a test particle.

3 Friedmann-Schwarzschild Crossed Metric

We define the metric by taking "a cross" between the Friedmann and Schwarzschild metrics by [4]

$$ds^2 = e^{\nu(t,\chi)} dt^2 - a^2(t)\left[e^{-\nu(t,\chi)} d\chi^2 + \sigma^2(\chi) d\Omega^2\right], \tag{12}$$

where $e^{\nu(t,\chi)} = [1 - 2M/a\sigma(\chi)]$, χ is the hyperspherical angle, $\sigma(\chi)$ is $\sinh\chi$, χ or $\sin\chi$ according as the model is open ($k = -1$), flat ($k = 0$) or closed ($k = 1$) and $a(t)$ is the corresponding scale parameter. Even in the flat case, when $\sigma^2(\chi) = \chi^2$, Eq.(12) does not reduce to Eq.(1) as the coefficient of $d\chi^2$ is time dependent here. Since the physical distance is being re-scaled this metric seems (relatively) more realistic than given by Eq.(1). Ofcourse neither is a realistic cosmological model. Since the conformal time-like Killing vector of the previous metric is no longer available the $c\psi N$-formalism cannot now be applied.

The $e\psi N$ force, for the flat Friedmann model, is simply

$$\left.\begin{array}{l} F_0 = \frac{3m\chi}{2(3\chi a_0^{2/3} t^{2/3} - 7Ma_0^{1/3})}\left[\frac{-4M^2 a_0^{1/3}}{9\chi t^{5/3}(a_0^{1/3} t^{2/3} - 2M)} + \frac{2(a_0^{2/3}\chi - 7Ma_0^{1/3} t^{-2/3})}{9\chi t^{1/3}}\right. \\ \qquad \left. + \frac{4M(3a_0^{2/3}\chi - 7Ma_0^{1/3} t^{-2/3})}{9\chi(a_0^{1/3} t\chi - 2Mt^{1/3})}\right], \\ F_1 = -\frac{mM}{a_0^{1/3} t^{2/3} \chi^2 (1 - 2M/a\chi)}, \quad F_2 = F_3 = 0. \end{array}\right\} \tag{13}$$

The time at which the repulsive force of the model inverts to an attractive force can be obtained by making $F_0 = 0$. It will be

$$t_I = \frac{(M/2\chi)^{2/3}}{a_0^{1/2}}. \tag{14}$$

We shall call this the "inversion time".

For the open Friedmann model (for sufficiently small values of t, for a given χ), the $e\psi N$ force, is

$$\left.\begin{array}{l} F_0 = -\frac{8m\sinh\chi\{(12t/a_0)^{-5/3}-\frac{1}{60}(a_0/12t)\}\{1-\frac{3}{20}(12t/a_0)^{2/3}\}}{3a_0\sinh\chi\{\frac{1}{2}(12t/a_0)^{2/3}+\frac{3}{40}(12t/a_0)^{4/3}\}-14M}\Big[\frac{12M^2}{a_0\sinh\chi(a\sinh\chi-2M)} \\ \quad \{1-\frac{3}{10}(12t/a_0)^{2/3}\}+\frac{7M}{a_0\sinh\chi}\{1+\frac{1}{30}(12t/a_0)^{2/3}\}\Big], \\ F_1 = -\frac{4mM(a_0/12t)^{2/3}}{a_0\sinh^2\chi(1-2M/a\sinh\chi)}\{1-\frac{3}{20}(12t/a_0)^{2/3}\}, \quad F_2 = F_3 = 0. \end{array}\right\} \tag{15}$$

The inversion time for the open Friedmann model will be

$$t_I = M/3\sinh\chi. \tag{16}$$

For the closed Friedmann model of the universe (for sufficiently small t, for the given χ), this takes the form

$$\left.\begin{array}{l} F_0 = -\frac{8m\sin\chi\{(12t/a_0)^{-5/3}+\frac{1}{60}(a_0/12t)\}\{1+\frac{3}{20}(12t/a_0)^{2/3}\}}{3a_0\sin\chi\{\frac{1}{2}(12t/a_0)^{2/3}-\frac{3}{40}(12t/a_0)^{4/3}\}-14M}\Big[\frac{12M^2}{a_0\sin\chi(a\sin\chi-2M)} \\ \quad \{1+\frac{3}{10}(12t/a_0)^{2/3}\}+\frac{7M}{a_0\sin\chi}\{1-\frac{1}{30}(12t/a_0)^{2/3}\}\Big], \\ F_1 = -\frac{4mM(a_0/12t)^{2/3}}{a_0\sin^2\chi(1-2M/a\sin\chi)}\{1+\frac{3}{20}(12t/a_0)^{2/3}\}, \quad F_2 = F_3 = 0. \end{array}\right\} \tag{17}$$

The inversion time for the closed model turns out to be

$$t_I = \frac{a_0}{2}\left[\pi/2 + 4M/3a_0\sin\chi - (8M/3a_0\sin\chi)^{1/2} - 1\right]. \tag{18}$$

It is to be noted that the $e\psi N$ force, for the first order, comes out to be equal for each of the Friedmann models of the universe. The time component of the $e\psi N$ force, in each of the Friedmann universe models, gives a measure of the change of the gravitational potential energy of the test particle. The spatial component represents a ψN force for the Schwarzschild metric, modulo a local Lorentz factor for the flat case. However, this component reduces for the early stages of the open Friedmann model and increases for the early stages of the closed Friedmann universe. Further, we see from Eqs.(15) and (17) that the magnitude of the time component of the $e\psi N$ force decreases for the early stages of the open Friedmann model while it increases for the

early stages of the closed Friedmann model. The fact that we get the usual Newtonian force, as happens for the Schwarzschild metric, shows that our metric does, infact, give the effect of a gravitating particle of mass m. This is in agreement with the already evaluated force for the Schwarzschild metric in the ψN and the $e\psi N$-formalisms.

The $e\psi N$ potential for the flat Friedmann model will be

$$\left.\begin{array}{l} p_0 = m\ln[\frac{3t(a_0^{1/3}t^{2/3}\chi-2M)^2}{2(a_0^{1/3}t^{2/3})^{12/7}(3a_0^{1/3}t^{2/3}\chi-2M)^{9/7}}], \\ V = m\ln(1-2M/a_0^{1/3}t^{2/3}\chi)^{1/2}. \end{array}\right\} \tag{19}$$

From here we note that p_0 tends to infinity as t approaches to zero. Infact, this gives the energy imparted to the test particle in the Friedmann model for the flat case.

It is worth mentioning that "in crossing" the two metrics the potentials have merely been "added". In principle each could have been modified by the other as well.

For the open Friedmann universe, the $e\psi N$ potential is

$$\left.\begin{array}{l} p_0 = m\ln[\frac{(a_0\sinh\chi)^{12/7}\{3a_0(\cosh\eta-1)\sinh\chi-4M\}^{9/7}\sinh\eta}{a_0\{a_0(\cosh\eta-1)\sinh\chi-4M\}^2(\cosh\eta-1)^{2/7}}], \\ V = m\ln[\frac{a_0(\cosh\eta-1)\sinh\chi-4M}{a_0(\cosh\eta-1)\sinh\chi}]^{1/2}. \end{array}\right\} \tag{20}$$

For the closed model of the Friedmann universe, the $e\psi N$ potential turns out to be

$$\left.\begin{array}{l} p_0 = m\ln[\frac{(a_0\sin\chi)^{12/7}\{3a_0(1-\cos\eta)\sin\chi-4M\}^{9/7}\sin\eta}{a_0\{a_0(1-\cos\eta)\sin\chi-4M\}^2(1-\cos\eta)^{2/7}}], \\ V = m\ln[\frac{a_0(1-\cos\eta)\sin\chi-4M}{a_0(1-\cos\eta)\sin\chi}]^{1/2}. \end{array}\right\} \tag{21}$$

The quantity p_0 yields the energy of the test particle for this crossed metric and the quantity V gives the usual ψN potential for the Schwarzschild metric, modulo a local Lorentz factor. It is worth noting that the time variaton and the usual Newtonian gravitational potential are acting together in this example. From here we see that the energy of the test particle becomes infinite at time $t = 0$ in each of the Friedmann models.

Notice that the force is repulsive at the early stages of the open Friemann model of the universe. But after a particular time, the attaractive force dominates the repulsive force. Thus there is an attractive component of the cosmological force in the expanding universe. This result coincides with the numerical results [7] which also indicate the dominance of the attractive force.

4 Friedmann-Kerr Crossed Metric

We have obtained some fundamentally new insights by considering the force four-vector for the Friedmann-Schwarzschild crossed metric. Further insights can be expected from the force four-vector by considering a more complicated metric than the Friedmann-Schwarzschild crossed metric, namely a non-static Kerr-like metric. This metric differs from the previous metric in that the Friedmann-Schwarzschild crossed metric does not have any conformal time-like Killing vector but it has a conformal time-like Killing vector. This metric has been defined [9] by multiplying the spatial part by a time factor. We shall call this metric the Friedmann-Kerr crossed metric. This has the following form

$$\left.\begin{array}{l} ds^2 = (1-2Mr/R^2)dt^2 - a^2(t)[(R^2/J)dr^2 + R^2 d\theta^2 + (P\sin^2\theta/R^2)d\phi^2] \\ \quad +\{2Mrba(t)\sin^2\theta/R^2\}dtd\phi, \end{array}\right\} \tag{22}$$

where

$$R^2 = r^2 + b^2\cos^2\theta, \quad J = r^2 - 2Mr + b^2, \quad P = (r^2+b^2)^2 - Jb^2\sin^2\theta \tag{23}$$

and $b(= s/M)$ is the spin or angular momentum per unit mass of the balck hole. The metric coefficients are given by

$$\left.\begin{array}{l} g_{00} = 1 - 2Mr/R^2, \quad g_{11} = -a^2R^2/J, \quad g_{22} = -a^2R^2, \\ g_{33} = -a^2P\sin^2\theta/R^2, \quad g_{03} = g_{30} = Mrab\sin^2\theta/R^2. \end{array}\right\} \tag{24}$$

Under suitable coordinate transformations [6], the off-diagonal elements vanish and we are left with the following metric coefficients

$$\left.\begin{array}{l} g_{00} = g_{00} - g^{33}g_{03}^2 = (1-2M/r)(1+4M^2b^2r^2\sin^2\theta/R^4J), \\ g_{11} = -a^2R^2/J, \quad g_{22} = -a^2R^2, \quad g_{33} = -a^2P\sin^2\theta/R^2. \end{array}\right\} \tag{25}$$

The $e\psi N$ force, for the flat Friedmann model of the universe $(k=0)$ is

$$\left.\begin{array}{l}
F_0 = m/3t, \\
F_1 = \frac{m}{JR^2(R^2-2Mr)(R^4J+4M^2b^2r^2\sin^2\theta)}[MR^4J^2(r^2-b^2\cos^2\theta) \\
\quad +4M^2b^2\sin^2\theta\{R^2J(2r^3-3Mr^2+rb^2\cos^2\theta) \\
\quad -r^2(R^2-2Mr)(3rJ+(r-M)R^2)\}], \\
F_2 = \frac{mMrb^2\sin 2\theta}{R^2(R^2-2Mr)(R^4J+4M^2b^2r^2\sin^2\theta)}[R^4J-2Mr\{(R^2-2Mr) \\
\quad (r^2+b^2+2b^2\sin^2\theta)-b^2\sin^2\theta\}], \\
F_3 = 0.
\end{array}\right\} \qquad (26)$$

Notice that the zero component coincides with that of the zero component of the flat Friedmann model [4]. It gives a rate of change of energy which approaches infinity at the very early stages of the Friedmann universe and goes to zero as t tends to infinity [4]. It is worth noting that the spatial components of the $e\psi N$ force F_1, F_2 and F_3 are just the ψN force for the Kerr metric for a special choice of geodesics [9,10], modulo a local Lorentz factor.

The $e\psi N$ "potentials" for this spacetime turn out to be

$$\left.\begin{array}{l}
U = -m\ln(t/T) - \frac{1}{2}m\ln[(R^2-2Mr)(R^4J+4M^2b^2r^2\sin^2\theta)/R^6J], \\
V = \frac{1}{2}m\ln[(R^2-2Mr)(R^4J+4M^2b^2r^2\sin^2\theta)/R^6J].
\end{array}\right\} \qquad (27)$$

Here if we add both these potentials the resultant will be the time component of the Friedmann metrics. Notice that the time component of the $e\psi N$ potential comes out to be the time component of the flat Friedmann model of the universe minus the radial and polar coordinate dependent term. The additional term occurs due to the g_{00} of the Kerr metric. Thus the potentials have merely been "added" when we "cross" the two metrics as in the case of the previous metric.

The expressions for the $e\psi N$ force and the $e\psi N$ potential are comprehensible and help us in understanding the energy of the test particle. We note that the spatial component of the $e\psi N$ potential reduces to the usual ψN potential of the Kerr metric for a special choice of geodesics [9,10], modulo a local Lorentz factor. The sum of both these potentials gives the potential energy imparted to the test particle. This energy goes to infinity for $t = 0$ as required for the Friedmann models of the universe.

5 Conclusion

We have constructed a cross model which gives an effect of gravitational source in all the Friedmann models of the universe. We then applied the $e\psi N$-formalism to this crossed metric so as to obtain the physical effects of forces. The spatial component of the force four-vector and the scalar quantity V give the Newtonian gravitational force and potential respectively for each of the Friedmann universes. This shows that the effect of gravitational source in an evolving universe. We have seen that the inversion time is different for each of the Friedmann models. Further we attempted to use the Kerr metric instead of Schwarzschild metric only for the flat case. In this case the time component just gives the energy imparted to a test particle for the flat Friedmann universe. We get physically acceptable effects in terms of forces and energy in each of the Friedmann models. The problem of constructing a new metric (i.e. cross between Kerr and Friedmann models) for each of the Friedmann metrics and then applying the $e\psi N$-formalism to it remains open.

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